Lecture 9 : Finitely generated groups

[2 weeks left on groups...] §36-38 of [R1]

A group G is *finitely generated* (f.g.) when there is a finite subset $S \subseteq G$ with $\langle S \rangle = G$.

Ex: $|G| < \infty$ as take $S = G$.

Ex: For $|S| < \infty$, the free group $F(S)$.

Ex: If G is f.g., $N \trianglelefteq G$, then G/N is f.g.

Non Ex: Any uncountable group, e.g. $(\mathbb{R}, +)$.

Non Ex: $F = F(T)$ where $|T| = \infty$.

Pf: Suppose $S \subseteq F$ is finite. Then only finitely many $t \in T$ appear in the reduced words for elts of S. As mult in F never introduces a letter, any $t' \in T$ not appearing in S is not in $\langle S \rangle$. So $\langle S \rangle \neq F$. ▨

Warning: A subgp of a f.g. G need not be f.g!

Let $G = F(a, b)$ and $H = \langle a^n b a^{-n} \text{ for } n \in \mathbb{Z} \rangle$

<u>Claim</u>: H is not f.g.

<u>Idea</u>: Set $R = F(\{X_n \mid n \in \mathbb{Z}\})$ and consider the homom $\phi : R \twoheadrightarrow H$ where $X_n \mapsto a^n b a^{-n}$. (symbol)

If ϕ is injective, have $H \cong R$ which is not f.g.

Suppose $r \in R$ is $X_{n_1}^{e_1} X_{n_2}^{e_2} \cdots X_{n_\ell}^{e_\ell}$ with $|e_i| > 0$ and $n_i \neq n_{i+1}$ for all i. Then $\phi(X_{n_i}^{e_i}) = a^{n_i} b^{e_i} a^{-n_i}$

and $\phi(r) = a^{n_1} b^{e_1} a^{n_2 - n_1} b^{e_2} a^{n_3 - n_2} \cdots a^{n_\ell - n_{\ell-1}} b^{e_\ell} a^{-n_\ell}$

where the latter is a reduced word. So $\ker(\phi) = \{e\}$.

<u>Fun fact</u>: Any subgp of a free group is free.

[Working towards subgps of f.g. <u>abelian</u> groups are f.g.]

A group G has the <u>ascending chain condition (acc)</u> when $\forall$ subgps $\{H_i\}_{i=1}^{\infty}$, where $H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots$ $\exists n$ such that $H_i = H$, for all $i \geq n$.

<u>Non Ex</u>: $G = F(\{a_k \mid k \in \mathbb{Z}_{>0}\})$ and

$$H_i = \langle a_1, a_2, \ldots, a_i \rangle.$$

<u>Thm</u>: G has the acc $\Longleftrightarrow$ all subgps of G are f.g.

<u>Pf</u>: ($\Rightarrow$) (Skip for time?) Suppose $H \leq G$ is <u>not</u> f.g.
Set $H_0 = \{e\}$. Inductively, pick $h_{i+1} \in H \setminus H_i$ and define $H_{i+1} := \langle h_1, \ldots, h_{i+1} \rangle$. Then the H_i violate the a.c.c. as $H_0 \subsetneq H_1 \subsetneq H_2 \subsetneq H_3 \subsetneq \cdots$

($\Leftarrow$) Suppose $H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots$ is a chain of subgps. Set $H = \bigcup_{i=1}^{\infty} H_i \leq G$ which must be $\langle x_1, \ldots, x_n \rangle$ for some $x_i \in H$. Pick m so that all x_i are in H_m, which gives $H = H_m$ as needed. ▨

<u>Thm</u>: Suppose $N \trianglelefteq G$. Then G has acc $\Longleftrightarrow$ N and G/N have the acc.

<u>Pf</u>: ($\Rightarrow$) Exercise.
($\Leftarrow$) Suppose $H_1 \leq H_2 \leq H_3 \leq \cdots \leq G$. Consider the chains $H_1 \cap N \leq H_2 \cap N \leq \cdots \leq N$ and

$$\phi(H_1) \leq \phi(H_2) \leq \cdots \leq G/N$$

where $\phi : G \to G/N$ is the quo. hom.

By hypothesis, $\exists\, m$ such that $H_k \cap N \overset{1}{=} H_m \cap N$ and $\phi(H_k) \overset{2}{=} \phi(H_m)$ for all $k \geq m$. Then $H_k = H_m$ for all $k \geq m$ since if $x \in H_k$ by 2 there is $y \in H_m$ with $\phi(x) = \phi(y) \Rightarrow \phi(y^{-1}x) = e$ and $y^{-1}x \in H_k$ so $y^{-1}x \in H_k \cap N \Rightarrow y^{-1}x \in H_m$ by 1 $\Rightarrow x = y \cdot (y^{-1}x) \in H_m$ as needed. ▨

Expect to end here

Thm: Every f.g. abelian group has the a.c.c.

Pf: We induct on the size n of a gen set for G.

$n = 0$: $G = \{e\}$, so done.

$n = 1$: G cyclic, easy as we know all subgps.

General case: Suppose $G = \langle a_1, \ldots, a_n \rangle$. Set $N = \langle a_1, a_2, \ldots, a_{n-1} \rangle$ which is normal since G is abelian. By induction, N has the acc as does $G/N = \langle \pi(a_n) \rangle$. Now apply the last theorem. ▨