Harmonic 1-forms on hyperbolic 3-manifolds: connections and computations

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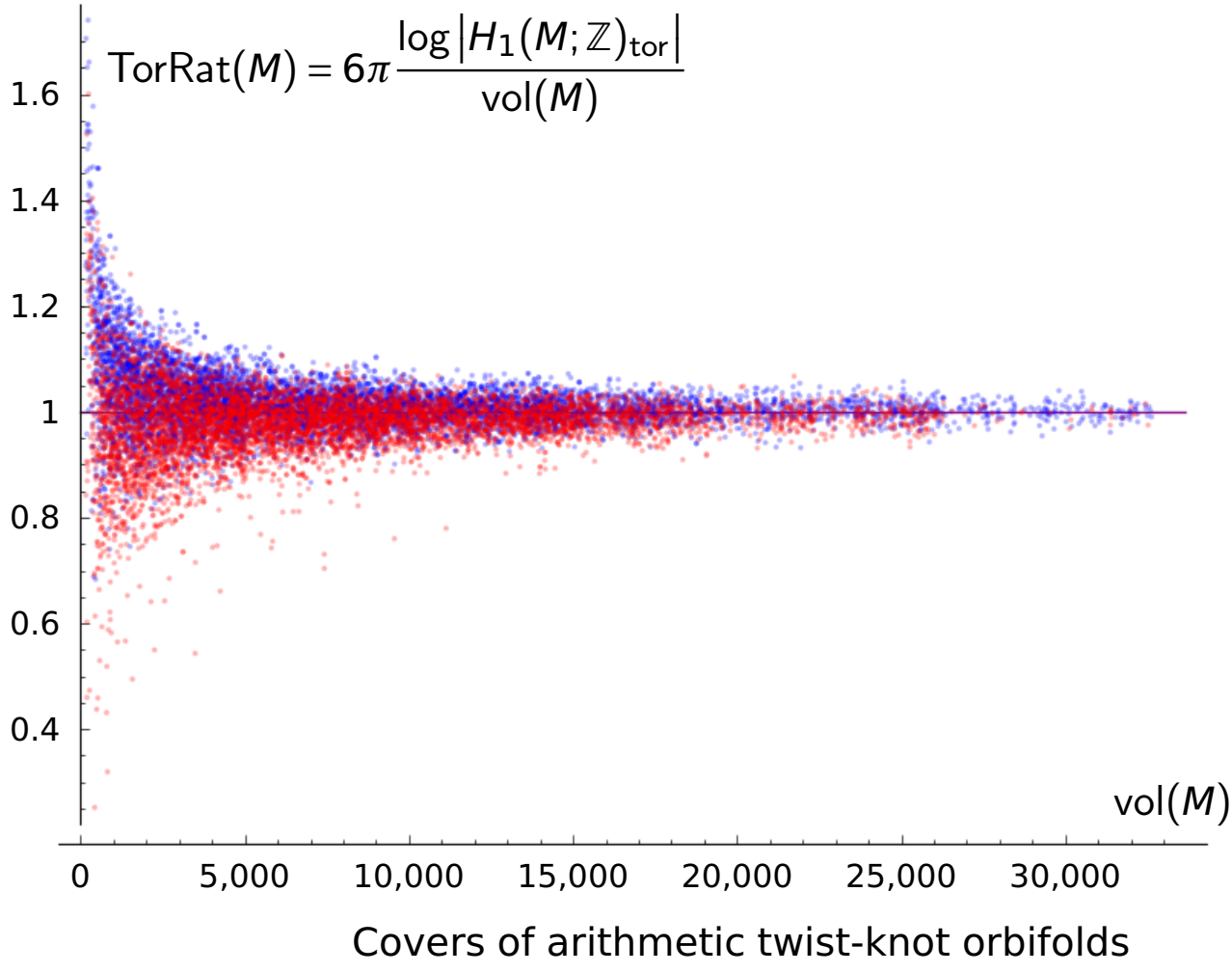
Slides posted at:
https://dunfield.info/slides/msri2020.pdf

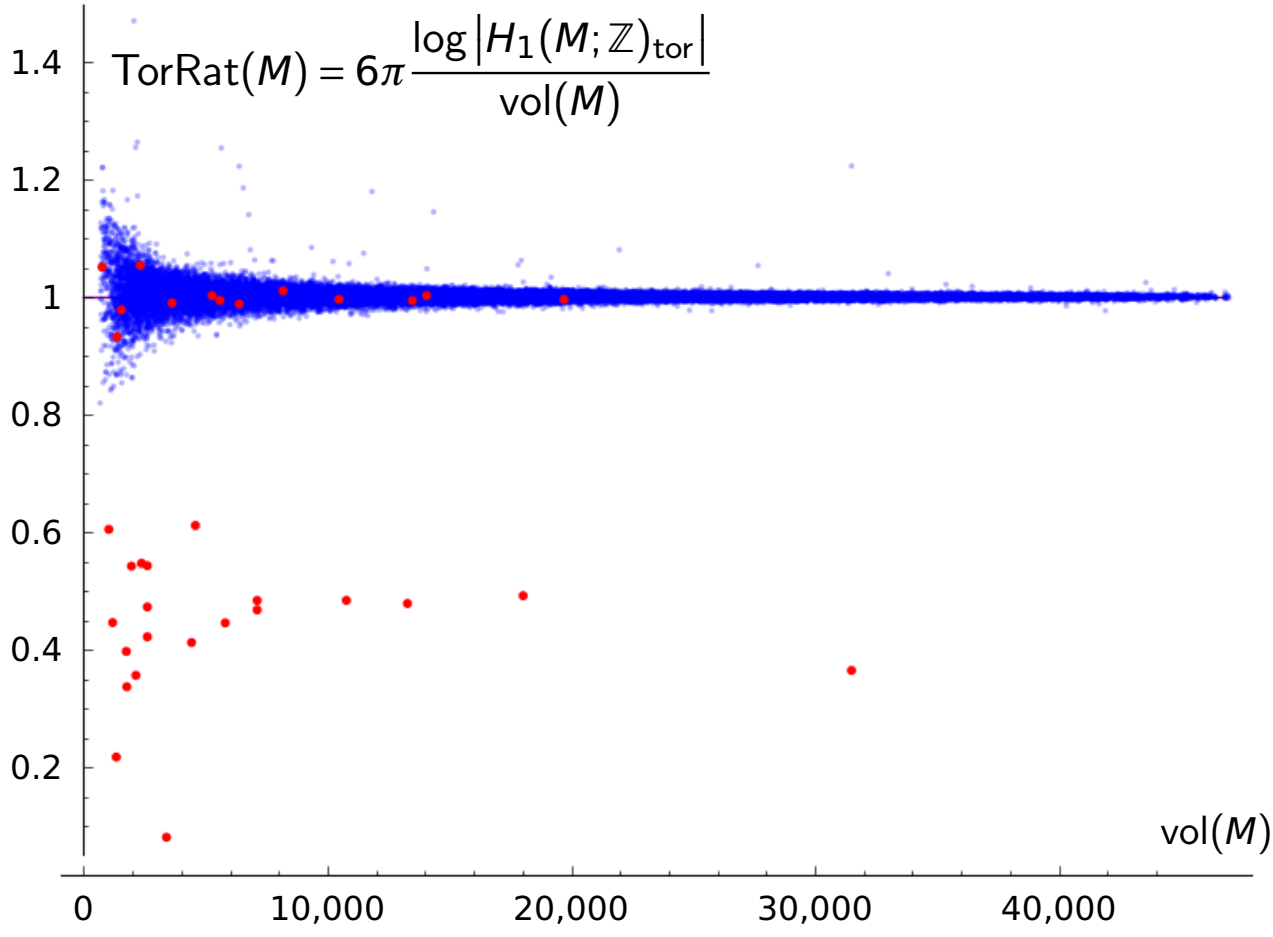
Conv: M^3 closed orient. hyperbolic

Conj [Bergeron-Venkatesh, Lê, Lück]
For a tower of congruence covers
$M = M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow M_4 \leftarrow \cdots$
where M is arithmetic and
$\operatorname{inj} M_n \to \infty$, one has:

$$\lim_{n\to\infty} \frac{\log\left|H_1(M_n;\mathbb{Z})_{\text{tor}}\right|}{\operatorname{vol}(M_n)} = \frac{1}{6\pi}$$

Qs. Evidence? Why? What if M is not arithmetic?


$\mathrm{TorRat}(M) = 6\pi \frac{\log |H_1(M;\mathbb{Z})_{\mathrm{tor}}|}{\mathrm{vol}(M)}$
1.6
1.4
1.2
1
0.8
0.6
0.4
$\mathrm{vol}(M)$
0
5,000
10,000
15,000
20,000
25,000
30,000
Covers of arithmetic twist-knot orbifolds



Covers of nonarithmetic twist-knot orbifolds

Ray-Singer analytic torsion:

$$\tau(N) = \frac{1}{2}\sum_{k=0}^{\dim N}(-1)^k \cdot k \cdot \log\left(\det{}'(\Delta_k)\right)$$

[Cheeger-Müller] For N^3, τ is:

$$\log(\mathrm{tor}(N)) - \log(\mathrm{vol}(N)) + 2\log(R^1(N))$$

where $\mathrm{tor}(N) = \left|H_1(N;\mathbb{Z})_{\mathrm{tor}}\right|$ and

$$R^1(N) = \mathrm{vol}\left(H^1(N;\mathbb{R})\Big/H^1(N;\mathbb{Z})\right)$$

and the volume form comes from $\|\cdot\|_{L^2}$ on $H^1(N;\mathbb{R})$.

[Bergeron-Şengün-Venkatesh 2016]

Suppose M_n is a tower of cong. covers of M with $\mathrm{inj}\, M_n \to \infty$.

Conj 1: Whether or not M is arith: $\tau(M_n)/vol(M_n) \to \tau^{(2)}(\mathbb{H}^3) = \frac{1}{6\pi}$.

Conj 2: When M is cong. arith $R^1(M_n) \leq \mathrm{vol}(M_n)^C$ where $C = C(M)$.

If both hold, then $\frac{\log(\mathrm{tor}(M_n))}{\mathrm{vol}(M_n)} \to \frac{1}{6\pi}$

[Brock-D] $\exists\, M_n \xrightarrow{BS} \mathbb{H}^3$ with all $H_1(M_n;\mathbb{Z}) = 0$ so $\tau(M_n)/\mathrm{vol}(M_n) \to 0$.

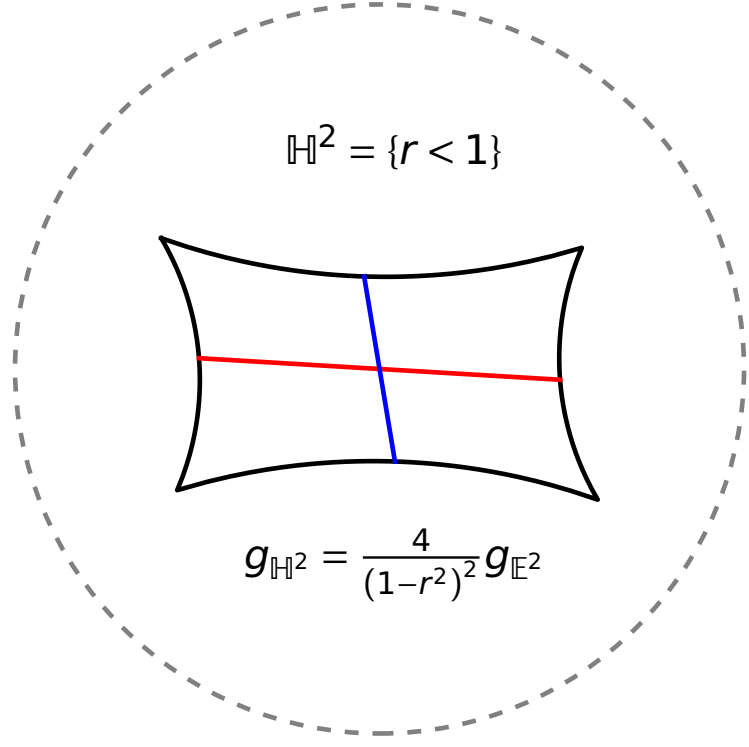
[Brock-D] $\exists\, M_n$ with $\mathrm{vol}(M_n) \to \infty$, $\mathrm{inj}(M_n) \geq \epsilon > 0$ and $R^1(M) \geq C^{\mathrm{vol}(M)}$ for some $C > 1$.

Harmonic norm:

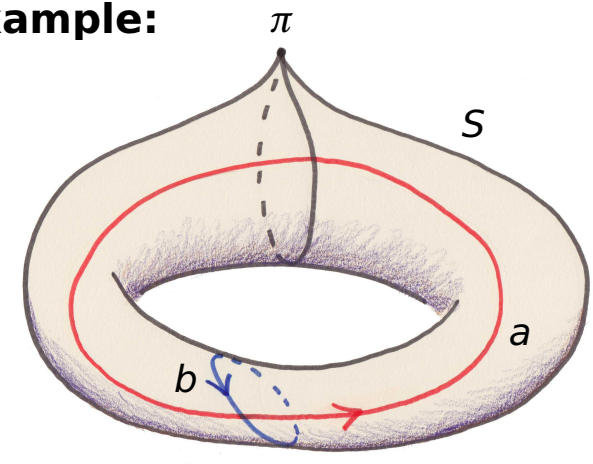
$\langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta$ for $\alpha, \beta \in \Omega^1(M)$.

$\phi \in H^1(M;\mathbb{R})$ has a unique *harmonic rep.* minimizing $\|\alpha\|_{L^2} = \sqrt{\langle \alpha, \alpha \rangle}$.

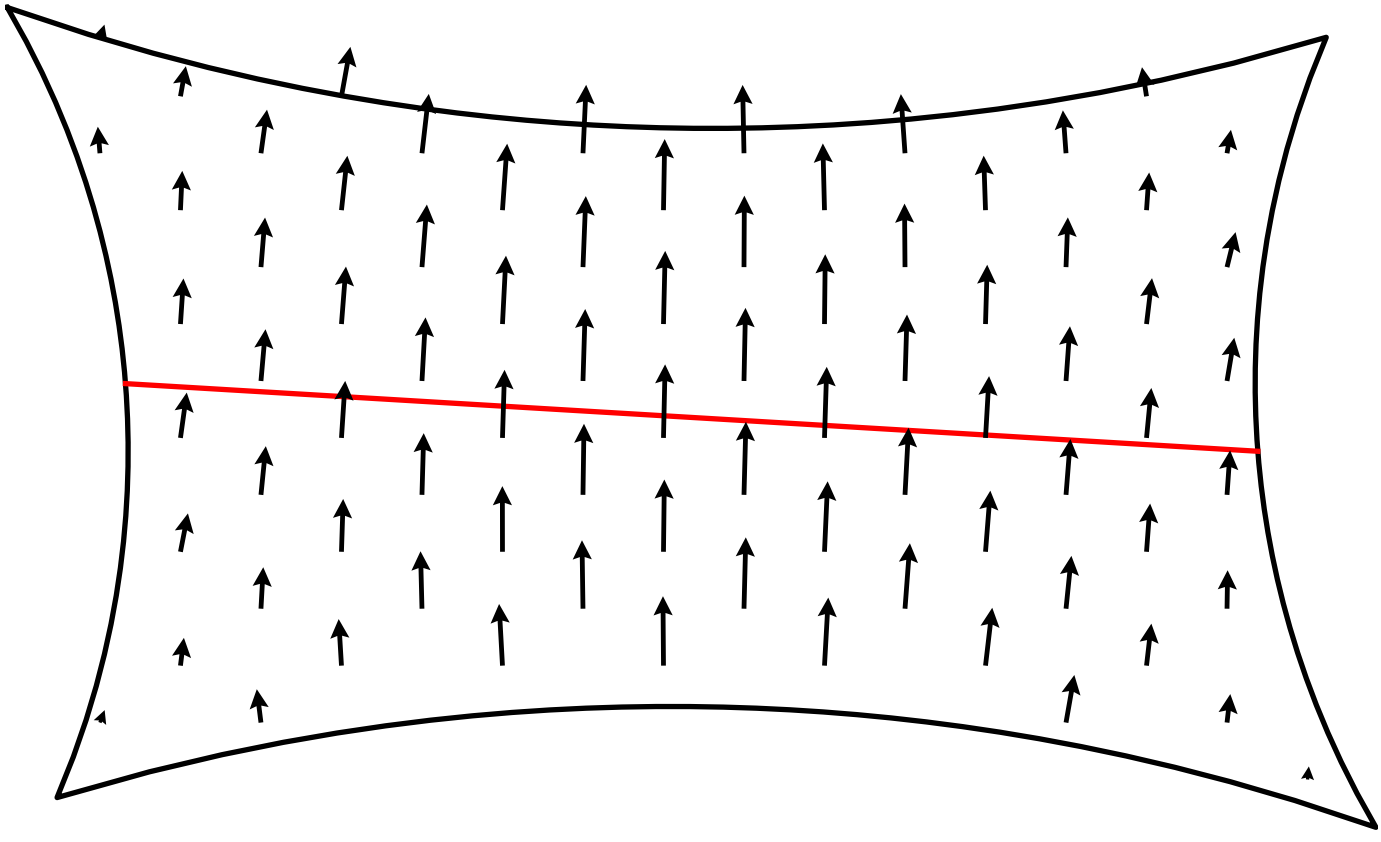
So $H^1(M;\mathbb{R}) \cong \ker(\Delta_1)$ inherits $\| \cdot \|_{L^2}$.



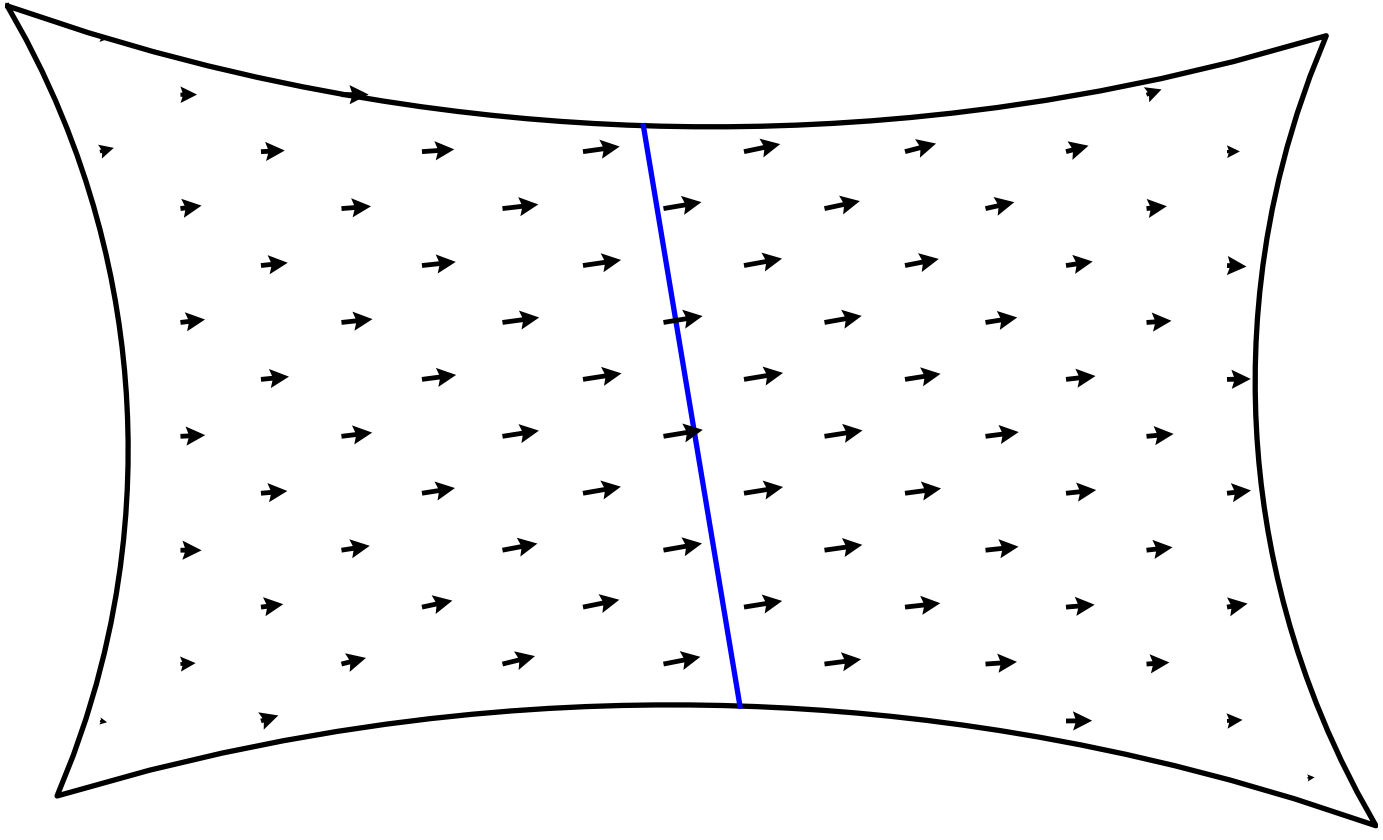
Example:

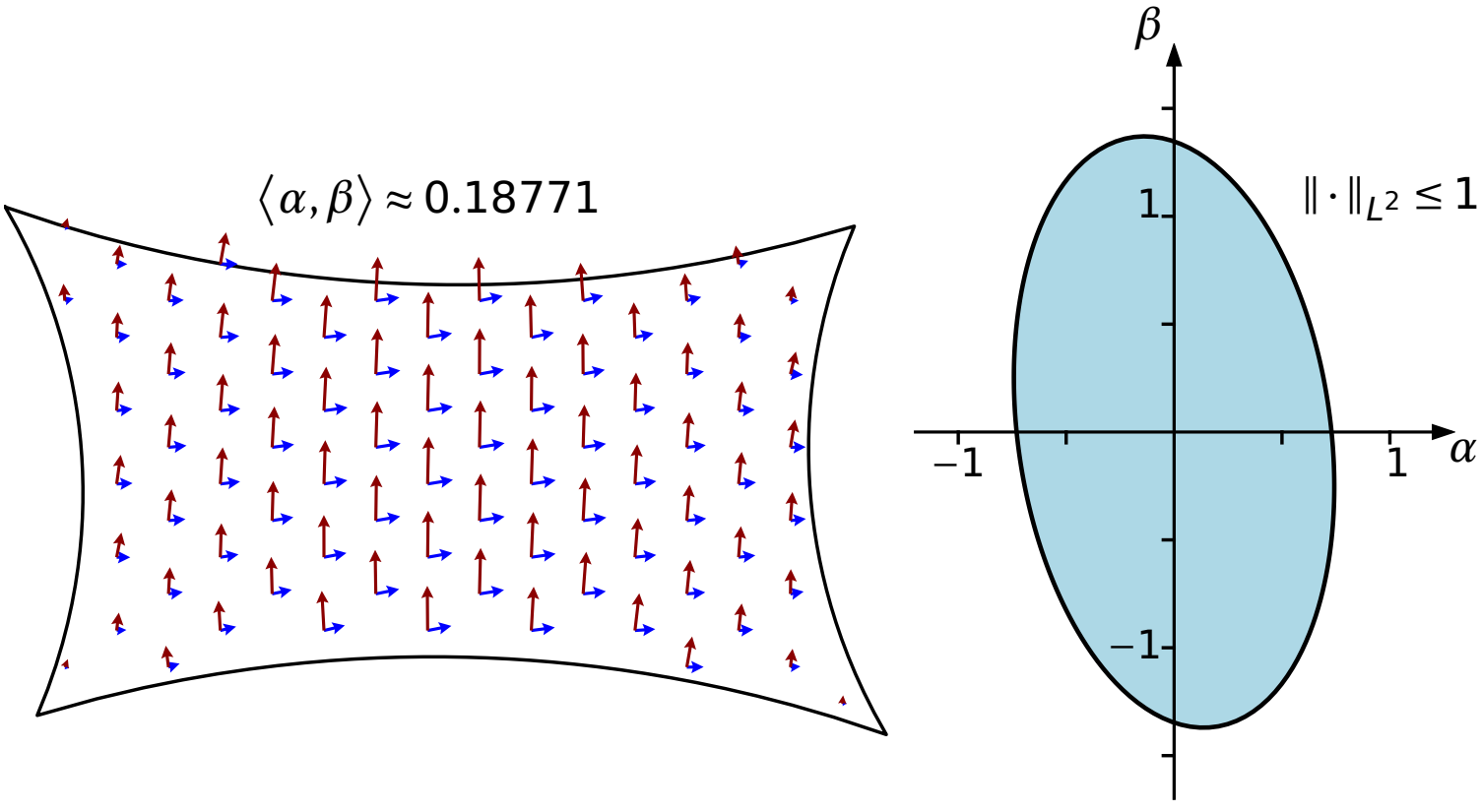


The Poincaré dual α of a has $\|\alpha\|_{L^2} \approx 1.37052$.

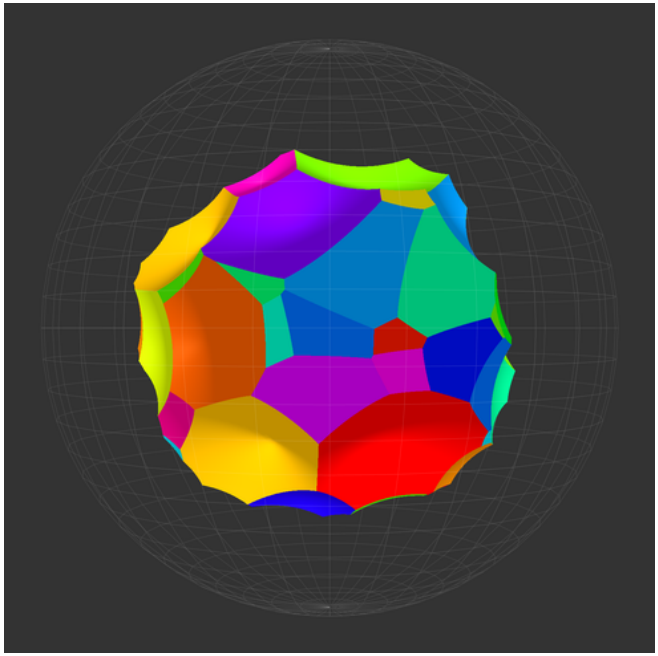


The Poincaré dual β of b has $\|\beta\|_{L^2} \approx 0.74239$




$\langle \alpha, \beta \rangle \approx 0.18771$
β
α
$\|\cdot\|_{L^2} \leq 1$
1
−1
1
−1

Input: Dirichlet domain D for M with face pairings in $\mathrm{Isom}^+(\mathbb{H}^3)$ and $c \in C_2(\bar{D};\mathbb{Z})$.



Output: Approx for the harmonic α dual to $[c]$.

Method of particular solutions

[Moler-Payne 1968; Betcke-Trefethen 2005; Barnett 2009; Strohmaier-Uski 2013]

Idea: Have $\widetilde{\alpha} = \pi^{-1}(\alpha) = df$ on $\mathbb{H}^3$. Expand about the center of D in a basis $\Phi_{\ell m}$ for harmonic functions:

$$f = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} \Phi_{\ell m}$$

Truncate ($\ell \leq L$) and use matching conditions on ∂D to solve for $a_{\ell m}$.

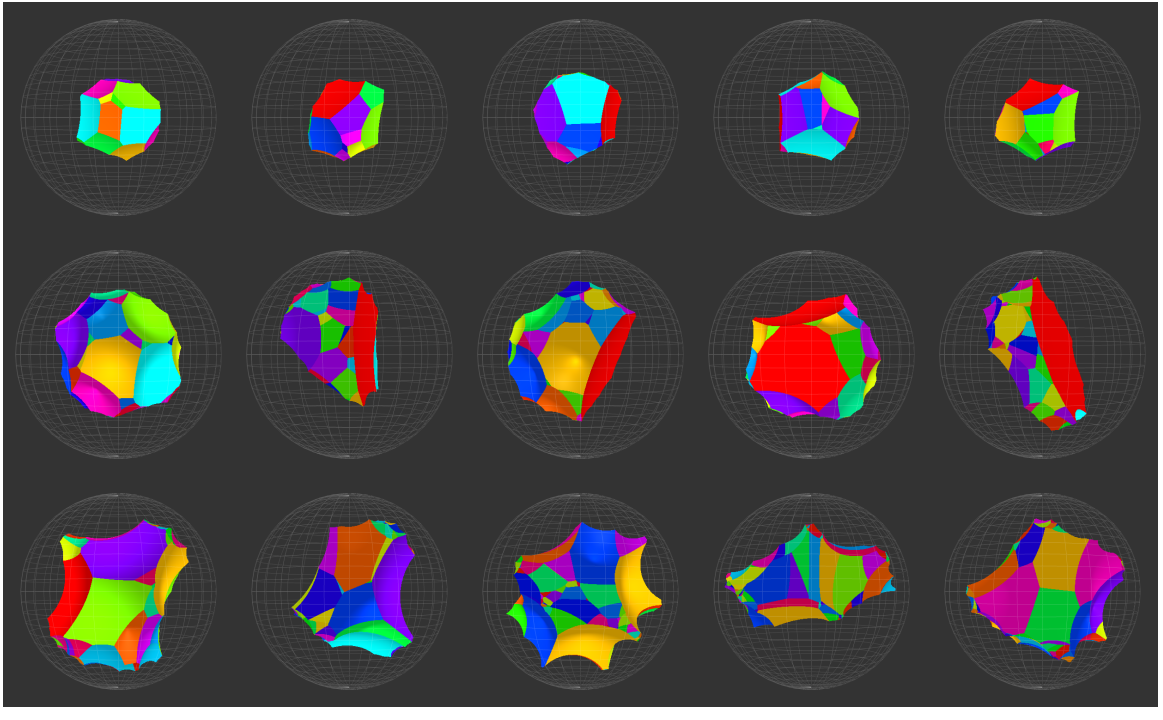
[D-Hirani] If $\bar{f}$ on $\mathbb{H}^3$ is harm, then

$$\left\| \widetilde{\alpha} - d\bar{f} \right\|_{L^2(D)} \leq 2\left(1 + \frac{1}{\lambda_1^0(M)}\right) E(\bar{f})$$

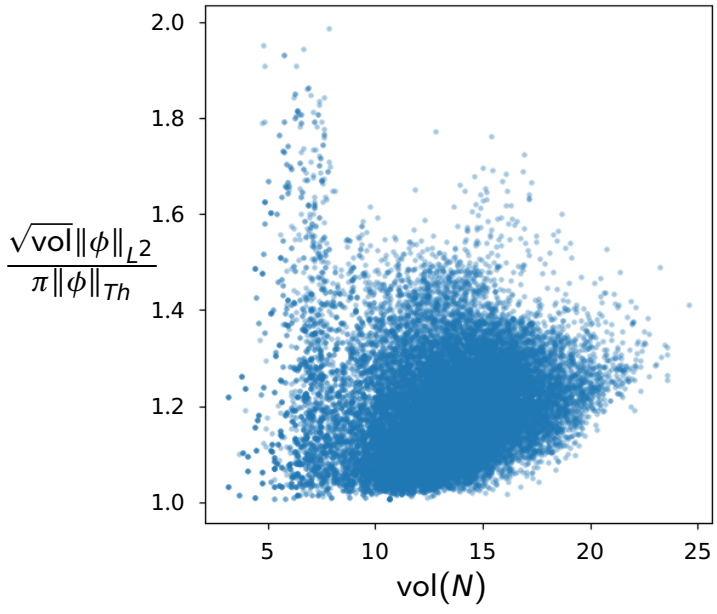
where $E(\bar{f})$ measures the failure of $d\bar{f}$ to be equivariant.

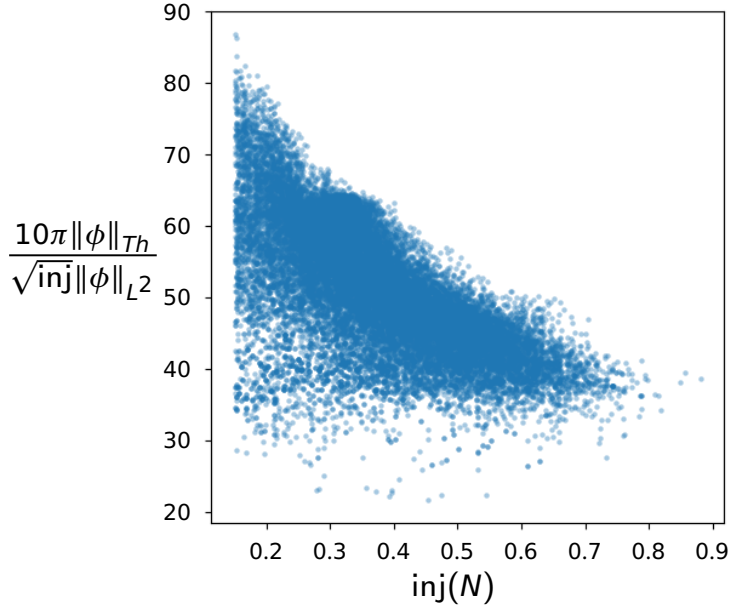
Sample: 24,735 mflds with $b_1 = 1$; $\mathrm{vol} \in [3.1, 24.6]$ and $\mathrm{inj} \in [0.15, 0.89]$. Computed $\|\phi\|_{L^2}$ for a gen ϕ of $H^1(M;\mathbb{Z})$ using $\ell \leq 50$ (2600 terms).

Ex: $M = m160(3,1) = \mathrm{Map}(S_2 : aabCDE)$ has $\mathrm{vol}(M) \approx 3.16633332124$ and $\|\phi\|_{L^2} \approx 3.646480936437$ and $\tau(M) \approx 2.5335632022$.



[Brock-D 2017; BŞV 2016] $\frac{\pi}{\sqrt{\mathrm{vol}(N)}}\|\cdot\|_{Th} \le \|\cdot\|_{L^2} \le \frac{10\pi}{\sqrt{\mathrm{inj}(N)}}\|\cdot\|_{Th}$





[X. Han 2020] $\frac{\pi}{\sqrt{\mathrm{vol}(N)}}\|\cdot\|_{Th} < \|\cdot\|_{L^2}$ and generalized both to finite-vol.